Supercuspidal Representations of GL_2.

Let F be a non-archimedean local field.

Definition: A supercuspidal representation of $G = GL_2(F)$ is a smooth G-representation (π, V) such that:

(i) (π, V) is irreducible.

(ii) if $(\pi^\vee, V^\vee)$ is the smooth dual of (π, V) and $v \in V$ and $v^\vee \in V^\vee$ then the "Matrix coefficient"

$$\gamma_{v \otimes v^\vee} : G \longrightarrow \mathbb{C}, \quad g \mapsto \langle \pi(g)v, v^\vee \rangle$$

is compactly supported modulo the centre Z of G.

Let $\mathcal{U} \leqslant G$ be an open subgroup such that $\mathcal{U}/Z$ is compact. Let $(\mathfrak{z}, W)$ be a smooth representation of $\mathcal{U}$.

Definition: The compact induction $c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z}$ is the G-representation (X, Σ) where

$$X = \left\{ \begin{array}{l} \text{locally constant functions} \\ f : G \longrightarrow W \end{array} : \begin{array}{l} \text{if } k \in \mathcal{U} \text{ and } g \in G \text{ then} \\ f(kg) = \mathfrak{z}(k).f(g) \text{ and} \\ f \text{ is compactly supported} \\ \text{modulo } \mathcal{U} \end{array} \right\}$$

and $\Sigma : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(X)$ "right translation"

$$g \mapsto \Sigma(g) : \Sigma(g)f(x) = f(xg)$$

where $f \in X$, $x \in G$.

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Notation: If $g \in G$ write (τ^g, W) for the $g^{-1}\mathcal{U}g$ representation $g^{-1}\mathcal{U}g \longrightarrow GL(W)$, $k \mapsto \tau(gkg^{-1})$.

Say $g \in G$ intertwines τ if

$$\mathrm{Hom}_{\mathcal{U}\cap g^{-1}\mathcal{U}g}\left(\mathrm{Res}^{g^{-1}\mathcal{U}g}_{\mathcal{U}\cap g^{-1}\mathcal{U}g}\tau^g, \mathrm{Res}^{\mathcal{U}}_{\mathcal{U}\cap g^{-1}\mathcal{U}g}\tau\right) \neq 0.$$

Theorem: If the following conditions hold:

(i) (τ, W) is irreducible.

(ii) If $g \in G$ intertwines τ then $g \in \mathcal{U}$.

then $c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\tau$ is supercuspidal and irreducible.

Sketch:

Step (1) Prove the implication: If $\pi := c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\tau$ is irreducible then π is supercuspidal.

How? Write $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\right\}$. One proves a coarse classification:

Theorem: (ρ, V) smooth irreducible.

Last time we proved a weaker version assuming admissibility

$$\left\{\begin{array}{l} \nexists\ \chi : B \to \mathbb{C}^{\times} \text{ a smooth character such} \\ \text{that } \rho \text{ is isomorphic to a subrepresentation} \\ \text{of } \mathrm{Ind}_B^G \chi \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \text{The matrix coefficients of } (\rho, V) \\ \text{are compactly supported modulo } Z \\ \text{i.e. } (\rho, V) \text{ is supercuspidal.} \end{array}\right\}$$

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This coarse classification affords one enough technical control to prove: If (ρ, V) is a smooth irreducible then (ρ, V) is admissible.

- Assume $\pi := c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G} \mathfrak{z}$ is irreducible. Then π is admissible.

We also understand $\pi^\vee$. If $\langle \cdot, \cdot \rangle : W \times W^\vee \longrightarrow \mathbb{C}$ is the canonical, non degenerate, $\mathcal{U}$-invariant pairing, then

$$\langle\langle \phi, \psi \rangle\rangle := \int_{\mathcal{U} \backslash G} \langle \phi(g), \psi(g) \rangle \, d_{\mathcal{U}} g$$

G-right invariant measure on $\mathcal{U} \backslash G$.

defines a non-degenerate, G-invariant pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G} \mathfrak{z} \times \mathrm{Ind}_{\mathcal{U}}^{G} \mathfrak{z}^\vee \longrightarrow \mathbb{C}.$$

So $\pi^\vee \simeq \mathrm{Ind}_{\mathcal{U}}^{G} \mathfrak{z}^\vee$. Can produce a non-zero compactly supported (modulo) Z matrix coefficient for π.

Take $w \in W$ and $w^\vee \in W^\vee$ such that $\langle w, w^\vee \rangle \neq 0$.

The matrix coefficient $\gamma_{\phi_w \otimes \phi_{w^\vee}}$ is non-zero and compactly supported where $\phi_w \in c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G} \mathfrak{z}$ and $\phi_{w^\vee} \in \mathrm{Ind}_{\mathcal{U}}^{G} \mathfrak{z}^\vee$ are

$$\phi_w(x) = \begin{cases} \mathfrak{z}(x).w, & x \in \mathcal{U} \\ 0, & x \notin \mathcal{U} \end{cases}, \qquad \phi_{w^\vee}(x) = \begin{cases} \mathfrak{z}^\vee(x) w^\vee, & x \in \mathcal{U} \\ 0, & x \notin \mathcal{U}. \end{cases}$$

So π is a smooth irreducible admissible representation and $\gamma_{\phi_w \otimes \phi_{w^\vee}}$ is a non-zero compactly supported (mod Z) matrix coefficient of π.

It turns out that: if (ρ, V) is a smooth irreducible admissible representation and $\exists\ (v, v^\vee) \in V \times V^\vee$ such that $\gamma_{v \otimes v^\vee}$ is non-zero and compactly supported modulo Z, then (ρ, V) is supercuspidal.

Why? (ρ, V) smooth irred. admissible $\Longrightarrow$ $V \otimes V^\vee$ smooth irred. $G \times G$-representation.

Jacobson Density Thm.

$\Rightarrow$ {The surjective $(G \times G)$-homomorphism

$$V \otimes V^\vee \longrightarrow C(\rho) := \mathbb{C}\text{-span}\{\gamma_{v \otimes v^\vee} : v \in V,\ v^\vee \in V^\vee\}$$
$$v \otimes v^\vee \longmapsto \gamma_{v \otimes v^\vee}$$

is an isomorphism.}

$\Rightarrow$ {$C(\rho)$ is irreducible as a $G \times G$-representation.}

$\Rightarrow$ {If $\exists (v, v^\vee) \in V \times V^\vee$ such that $\gamma_{v \otimes v^\vee}$ is non-zero and compactly supported (mod Z) then $\gamma_{v \otimes v^\vee}$ generates $C(\rho)$ as a $G \times G$-module and every matrix coefficient of ρ is compactly supported (modulo Z).}

"True even without intertwining condition"

This concludes the sketch of Step ①.

Step ② Show that $\pi := c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\, \tau$ is irreducible.

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The intertwining condition implies that (also need σ irreducible).

$$\dim_{\mathbb{C}} (\mathrm{End}_G(\pi)) = 1.$$

necessary for π irreducible.

Why? The σ-spherical Hecke algebra is

$$\mathcal{H}(G,\sigma) := \left\{ \begin{array}{l} \text{locally constant} \\ \text{functions} \\ f : G \to \mathrm{End}_{\mathbb{C}}(W) \end{array} : \begin{array}{l} f \text{ is compactly supported} \\ \text{modulo } Z \text{ and if } k_1, k_2 \in U \\ \text{and } g \in G \text{ then} \\ f(k_1 g k_2) = \sigma(k_1) \circ f(g) \circ \sigma(k_2) \end{array} \right\}$$

with convolution product

$$\phi_1 * \phi_2(g) = \int_{G/Z} \phi_1(x) \circ \phi_2(x^{-1}g)\, d_Z x, \quad (g \in G,\ \phi_1, \phi_2 \in \mathcal{H}(G,\sigma)).$$

Fixed Haar measure on G/Z

For $\phi \in \mathcal{H}(G,\sigma)$ and $f \in c\text{-}\mathrm{Ind}_U^G \sigma$ the

function $\phi * f(g) = \int_{G/Z} \phi(x)(f(x^{-1}g))\, d_Z x \quad (g \in G)$

is in $c\text{-}\mathrm{Ind}_U^G \sigma$.

The map

$$\begin{aligned} \mathcal{H}(G,\sigma) &\longrightarrow \mathrm{End}_G(\pi) \\ \phi &\longmapsto \phi * (-) \end{aligned}$$

is an isomorphism of $\mathbb{C}$-algebras.

One can check explicitly that "intertwining condition" plus "σ-irreducible" $\Longrightarrow \dim_{\mathbb{C}} \mathcal{H}(G,\sigma) = 1$.

(Every function is a scalar multiple of $f(g) = \begin{cases} 0, & g \notin U \\ \sigma(g), & g \in U. \end{cases}$)

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So $\dim_{\mathbb{C}}(\mathrm{End}_G(c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z})) = 1.$

If $X^{\mathfrak{z}}$ is the $\mathfrak{z}$-isotypic component of $c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z}$ then

$$\mathrm{Hom}_{\mathcal{U}}(W, X^{\mathfrak{z}}) = \mathrm{Hom}_{\mathcal{U}}(W, X) \simeq \mathrm{End}_G(c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z})$$

Frobenius Reciprocity.

So $W = X^{\mathfrak{z}}$. Let $Y \subseteq c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z}$ be a non-zero G-submodule.

then

$$0 \neq \mathrm{Hom}_G(Y, c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z}) \subseteq \mathrm{Hom}_G(Y, \mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z})$$
$$\simeq \mathrm{Hom}_{\mathcal{U}}(Y, \mathfrak{z})$$

So

$$0 \neq Y^{\mathfrak{z}} \subseteq X^{\mathfrak{z}} = W$$

and W irreducible $\Longrightarrow$ $Y^{\mathfrak{z}} = W$.

In particular $Y \supseteq W$. Since W generates $c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z}$ as a G-module, $Y = c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G}\,\mathfrak{z}$. $\blacksquare$ (sketch)

Remarks: (i) An element $g \in G$ intertwines $\mathfrak{z}$ if and only if every element in $\mathcal{U}g\mathcal{U}$ intertwines $\mathfrak{z}$.

(ii) If $\dim \mathfrak{z} = 1$, i.e $\mathfrak{z}$ is a character

$$\mathfrak{z} : \mathcal{U} \longrightarrow \mathbb{C}^{\times}$$

then $g \in G$ intertwines $\mathfrak{z}$ if and only if

$$\mathfrak{z}(k) = \mathfrak{z}(gkg^{-1}) \quad \forall\, k \in \mathcal{U} \cap g^{-1}\mathcal{U}g.$$

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Remark (cont.) (iii) Supercuspidal (as we've defined it) makes sense for the F-points of any connected reductive algebraic grp. Our main theorem holds in this setting also.

§ The simple supercuspidal for $SL_2(\mathbb{Q}_2)$.

The Iwahori subgroup $I \leq SL_2(\mathbb{Q}_2)$ is

$$I = \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in SL_2(\mathbb{Q}_2) : a,b,c,d \in \mathbb{Z}_2 \right\}$$

So I is a pro-2 group.

- I is compact, open, and contains the centre
$$Z_{SL_2} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}.$$

Let $\chi : I \longrightarrow \mathbb{C}^\times$ denote the character

$$\chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} = (-1)^{b+c \,(\mathrm{mod}\, 2)} \qquad a,b,c,d \in \mathbb{Z}_2, \; ad-2bc = 1.$$

Proposition: The compact induction $\pi := c\text{-}\mathrm{Ind}_I^{SL_2(\mathbb{Q}_2)}(\chi)$ is irreducible and supercuspidal.

Proof: By the intertwining criterion it suffices to show:

If $g \in SL_2(\mathbb{Q}_2)$ intertwines χ then $g \in I$. ✡

By Remark (i) it suffices to check ✡ on a system of representatives for the double coset space $~~I\backslash SL_2(\mathbb{Q}_2)~~/I$ $I\backslash SL_2(\mathbb{Q}_2)/I$.

"The Iwahori Decomposition"

The affine Weyl group of SL_2 is $W_{aff} := \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$.

W_{aff} indexes the double cosets $\{ I g I \mid g \in SL_2(\mathbb{Q}_2) \}$

by
$$W_{aff} \xrightarrow{\sim} I \backslash SL_2(\mathbb{Q}_2) / I$$
$$w = s_{i_1} \cdots s_{i_l} \longmapsto I n_w I$$

where $i_1, \dots, i_l \in \{0, 1\}$, $w = s_{i_1} \cdots s_{i_l}$ is a reduced word for w and $n_w = n_{i_1} \cdots n_{i_l}$ with $n_0 = \begin{pmatrix} 0 & 1/2 \\ -2 & 0 \end{pmatrix}$

and $n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

So $\{ n_w : w \in W_{aff} \}$ is a system of representatives for $I \backslash SL_2(\mathbb{Q}_2) / I$. It is enough to show:

If $w \in W_{aff}$ and $\chi^{n_w}(k) = \chi(k)$ for all $k \in n_w^{-1} I n_w \cap I$ then $w = 1$.

Remark (ii). $\chi^{n_w}(k) = \chi(n_w k n_w^{-1})$.

Suppose $\exists\ w \in W_{aff} - \{1\}$ such that if $k \in n_w^{-1} I n_w \cap I$ then $\chi^{n_w}(k) = \chi(k)$. We proceed to a contradiction.

"Affine Root Subgroups in SL_2"

Let Ψ denote the set of symbols
$$\Psi = \{ \alpha + n, -\alpha + n : n \in \mathbb{Z} \}$$

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For $\psi \in \Psi$ define the affine root subgroup.

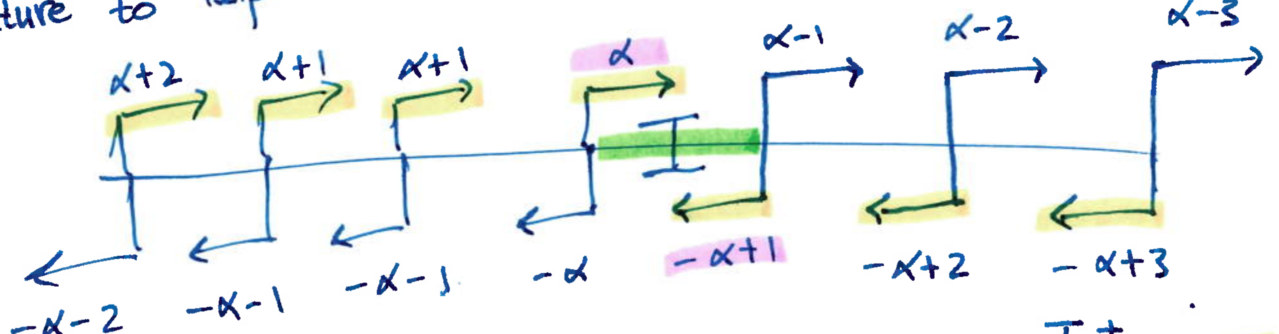
$$\mathcal{U}_\psi = \begin{cases} \left\{ \begin{pmatrix} 1 & 2^n x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}_2 \right\}, & \text{if } \psi = \alpha + n, \\ \left\{ \begin{pmatrix} 1 & 0 \\ 2^n x & 1 \end{pmatrix} : x \in \mathbb{Z}_2 \right\}, & \text{if } \psi = -\alpha + n. \end{cases}$$

Observations: Let $\psi \in \Psi$:

(i) $\mathcal{U}_\psi \leq I \iff \psi \in \Psi^+ := \{\alpha + n, -\alpha + n + 1 : n \in \mathbb{Z}_{\geq 0}\}$.

(ii) $\chi|_{\mathcal{U}_\psi} \neq 1 \iff \psi \in \Pi := \{\alpha, -\alpha + 1\}$.

Picture to help with notation:



$\Psi^+ =$ [yellow]

$\Pi =$ [pink]

(iii) W_{aff} acts on $\{\mathcal{U}_\psi : \psi \in \Psi\}$,

$$W_{aff} \times \{\mathcal{U}_\psi : \psi \in \Psi\} \longrightarrow \{\mathcal{U}_\psi : \psi \in \Psi\}$$

$$(w, \mathcal{U}_\psi) \longmapsto n_w \mathcal{U}_\psi n_w^{-1}.$$

"matrix calculation"

(iv) If $w \in W_{aff} - \{1\}$ then there exists $\psi' \in \Pi$ such that $n_w \mathcal{U}_{\psi'} n_w^{-1} \leq I$ and $n_w \mathcal{U}_{\psi'} n_w^{-1} \neq \mathcal{U}_\alpha$ and ~~or~~ $n_w \mathcal{U}_{\psi'} n_w^{-1} \neq \mathcal{U}_{1-\alpha}$ i.e $w.\psi' \in \Psi^+ - \Pi$.

Remark: (iv) is *the* essential property. It is generally true of affine root systems. It fails spectacularly for finite root systems.

Now we can finish the proof. Suppose $w \in W_{aff} - \{1\}$ is such that n_w intertwines χ. If $k \in n_w^{-1} I n_w \cap I$ then

$$\chi^{n_w}(k) = \chi(k). \quad (*)$$

Since $w \neq 1$, observation (iv) $\Longrightarrow \exists \psi \in \Pi$ such that $n_w U_\psi n_w^{-1} \leq I$ and $n_w U_\psi n_w^{-1} \notin \{U_\alpha, U_{1-\alpha}\}$. So by observation (ii) and (iii), $\chi|_{n_w U_\psi n_w^{-1}} = 1$.

So $\chi^{n_w}|_{U_\psi} = 1$.

As $U_\psi \leq I \cap n_w^{-1} I n_w$, the intertwing condition $(*)$ implies $\chi|_{U_\psi} = \chi^{n_w}|_{U_\psi}$. So $\chi|_{U_\psi} = 1$.

This contradicts observation (ii) since $\psi \in \Pi$.

So $w = 1$ and $n_w \in I$. This verifies the ~~intertwing~~ intertwining criterion. So $\pi := \text{c-Ind}_I^{SL_2(\mathbb{Q}_2)}(\chi)$ is irreducible and therefore supercuspidal. $\boxed{}$